A Discussion of Various Measures of Altitude

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It is common in dealing with airborne research data to encounter many different altitude terms. These include geometric altitude, GPS altitude, INS altitude, pressure altitude, geopotential height, and so on. Despite the nomenclature, there are only two altitude scales involved: *geometric altitude* and *geopotential altitude or height*. *Geometric altitude* is the scale we are most familiar with; it is what we would measure with a tape measure. Radiosondes and rawindsondes (collectively, RAOBs), on the other hand, generally report *geopotential height* -- a scale which relates height to gravitational equipotentials, or surfaces of constant gravitational potential energy per unit mass. Although geopotential height approximates geometric height, they are not the same. An important type of geopotential height is *pressure altitude*, which is based on a standard atmospheric model for temperature as a function of pressure. One particular model, the International Standard Atmosphere (ISA), is what all aircraft altimeters use to relate static pressure measurements on an aircraft to a corresponding *pressure altitude* scale. There are also a number of additional altitude terms related to flying airplanes such as true altitude, indicated altitude, absolute altitude and *density altitude*, which I discuss on another page for completeness.

Geometric and GPS Altitude

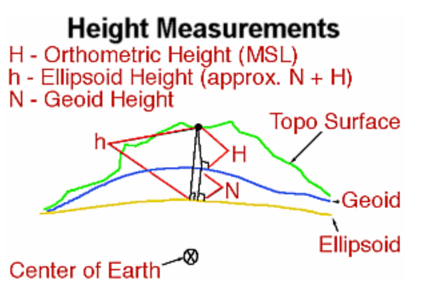
First we discuss geometric altitude, since it is the altitude scale that we are most familiar with. It is the height that we would measure with a tape measure in some length unit, say meters . Obviously it would be difficult to measure the height of an airplane with a tape measure, but other instruments are viable such as radar altimeters. However, altimeter measurements are relative to the local terrain, which could range from sea level to a mountain top. If airborne data are to be compared, it is necessary to introduce a reference surface to which all measurements are compared. This gets a little complicated because there are many possible reference surfaces, or *vertical datums* . Historically, mean sea level (MSL) has been used as the zero of elevation, and this is closely approximated by an equipotential surface for the earth's gravity field called the *geoid*. Although much smoother than the topographic surface of the earth, the *geoid* has significant vertical undulations due to the large scale distribution of the earth's mass. A much smoother reference surface is the *reference ellipsoid*, which is used to the approximate the shape of the earth. The *reference ellipsoid* is convenient because three dimensional coordinates (latitude, longitude, and altitude) are easily defined with respect to it.

This is a convenient point to introduce the **Global Positioning System** (GPS) of satellites. The *reference ellipsoid* for GPS is the so-called **WGS-84** (World Geodetic System 1984) system, which is defined by four parameters:

Semi-major axis (a):	6378.1370 km
Reciprocal of flattening (1/f):	298.257223563
Angular velocity of Earth (w):	$7292115.0 \times 10^{-11}$ rad/s
Earth's Gravitational Constant (Atmosphere incl.) (GM):	$3986004.418 \times 10^{8}$ m^3/s^2

Flattening (f) is a term used in geodesy instead of either the semi-minor axis or eccentricity; it is related to the semi-major and semi-minor axes by the expression: f = (a - b) / a. Using this expression, the semi-minor axis (b) is: 6356.7523 km, so the equatorial radius of the earth is 21.4 km greater than the polar radius. Note that the angular velocity is important because *gravity* has two components: the gravitational force between the Earth and another mass, and the centrifugal force due to the fact that the Earth is a non-inertial reference frame.

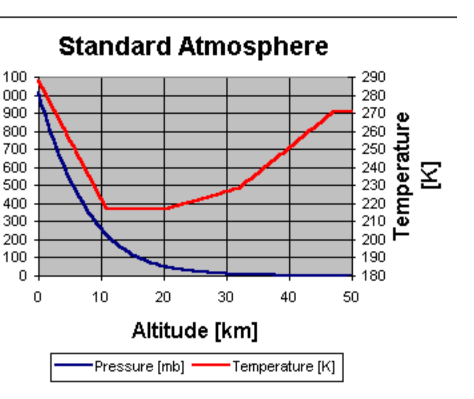
Using GPS satellite data, the WGS-84 system has been refined several times, most recently in 1999. However, because of the number of GPS receivers in use and the cost to replace them, the refinements are only used for satellite orbit determinations. Time corrections are transmitted via the GPS satellites which work with the **WGS-84** model software inside the GPS receivers. GPS measurements made relative to this *reference ellipsoid* are generally converted to different *vertical datums* by the software internal to the GPS receiver. This involves a geoid model, the current one being EGM96 (Earth Gravitational Model 1996). EGM96 is a spherical harmonic expansion of the gravitational potential of the Earth through degree (n) and order (m) 360, and is comprised of 130,317 coefficients!



At the risk of over-complicating this discussion, but in the interest of completeness, some more altitude terminology needs to be introduced. The height at or above the earth's surface can be measured with respect to the *reference ellipsoid* or the *geoid* , and these can differ by as much as 100 meters. While these reference surfaces differ in an absolute sense, they nevertheless lie on the same equipotential surface -- the surface defined by mean sea level. Referring to the figure to the right, the height normal to the *geoid* is called the *orthometric height* (H). The normal distance, which can be positive or negative, from the *reference ellipsoid* to the *geoid* is called *geoid height* (N). Finally the normal distance from the *reference ellipsoid* to some point above the earth's surface is termed the *ellipsoid height* , or *geodetic height* (h); it is the sum of the *geoid height* and the *orthometric height* . For further discussion go to the National Imagery and Mapping Agency or the National Geodetic Survey .

For the purpose of the present discussion, we are interested in both the *orthometric height* and the *ellipsoid height* ; the former is needed since it is in the direction of normal gravity measured when surveying, and the latter is needed because it is what a GPS measures.They are related by the excellent approximation expression: h = N + H.

Pressure Altitude



Allthough pressure altitude is a type of geopotential height, I discuss it separately because of it's importance in atmospheric research. In fact the initial motivation for this web page was to show other scientists how to relate pressure altitude and geometric altitude. The motivation was simple: some instruments like lidars measure distance on a geometric altitude scale, while others, like my Microwave Temperature Profiler (MTP), measure distance on a pressure altitude scale. The pressure altitude scale is based on the International Civil Aviation Organization's (ICAO) International Standard Atmosphere (ISA), which is the same as the **US Standard Atmosphere (1976)** to an altitude of 32 km. The US Standard atmosphere (1976) is an average, piece-wise continuous, mid-latitude temperature profile of the earth's atmosphere. It can be used to establish -- via the hydrostatic equation and the ideal gas law -- a relationship between pressure and pressure altitude, using *geopotential height*. It differs from "normal" geopotentail height in that it is based on a model and it assumes that the humidity is zero. The model seldom looks like the actual atmosphere a plane is flying in, and real atmopsheres never have zero humidity. The US Standard Atmosphere is one of several reference or standard atmospheres. I only use it instead of the ISA because when I started doing this it was easier for me to get my hands on information about it than ISA. My apologies to aviation purists.

Pressure altitude is used so that aircraft, which use static pressure to determine altitude, can agree upon what "altitude" they are flying at without having to continually update their altimeters with local pressure corrections. Technically, this is only true above 18,000 feet (FL180). Below this altitude in North America aircraft make local altimeter corrections to ensure that they are flying at the correct altitude. Using this definition for pressure altitude, a pilot can say "*I'm at Flight Level 330*." (that is 33,000 feet), instead of "*I'm at 262 hPa*." Pressure just isn't very intuitive since it's logarithmic with altitude, and it also decreases with altitude. In addition to pressure altitude, there are five additional altitude scales relevant to aviation.

Pressure altitude is also useful to scientists involved with radiative transfer through the atmosphere since the absorption is normally expressed in terms of pressure (and other parameters such as temperature, frequency, and humidity). So pressure altitude is a "natural" co-ordinate for such calculations. On the other hand, absorption coefficients are often expressed per unit geometric distance, so it is often necessary to convert between the different distance/altitude scales depending on the application.

Geopotential Height

Having described one type of geopotential height with the cart in front of the horse, we now describe how to calculate geopotential height. The gravitational potential energy (Φ) of a unit mass of anything is simply the integral from mean sea level (z = 0 meters) to the height of the mass (z = Z). It is given by the equation:

(1) $\Phi = \int_0^Z \gamma(z,\phi)\, dz$,

where $\gamma(z,\phi)$ is the normal gravity above the geoid. It is a function of both geometric altitude (z) and geodetic latitude (ϕ). Note that normal gravity is what would be measured by a plumb line and it includes contributions from both gravitational and centrifugal forces. Now while the geopotential (potential energy per unit mass) is useful for atmospheric dynamics studies (since it is a convenient way to compare meteorological data from different locations), it would be more convenient if it could be expressed as a height above the geoid. To this end, the geopotential (Φ) was divided by the normal gravity (γ_{45}) at a latitude of 45 degrees to obtain the geopotential height scale:

(2) $H(Z,\phi) = \frac{\Phi(Z,\phi)}{\gamma_{45}} = \frac{1}{\gamma_{45}} \cdot \int_0^Z \gamma(z,\phi)\, dz$

A latitude of 45 degrees was chosen because it was the latitude used by the World Meteorological Organization (WMO) to calibrate barometers. Also, since surface gravity is greatest at the poles and least at the equator, this "splits the difference" and results in geopotential height being close to geometric height at mid-latitudes, since only the weak altitude dependence of $\gamma(z,\phi)$ remains. The difference between geometric altitude and geopotential height can be significant (~120 meters) near the equator at ER-2 cruise altitudes (20 km). Also, since $\gamma(z,\phi)$ decreases with height (except near the poles), the geopotential height is generally less than the geometric height. The original definition of γ_{45} was 9.8 m/s^2; however, this was changed in the US in 1935 to the current value: γ_{45}=9.80665 m/s^2. Finally, the earth's gravity model has been modified slightly since this most recent definition of γ_{45}, so γ_{45} is now the gravity at 45.542 degrees instead of 45.0 degrees. However, the definition of the geopotential height has not changed.

Historically, the motivation for introducing the concept of *geopotential height* was to come up with a scheme by which the pressure, temperature, and relative humidity measured by radiosondes around the world could be compared on a common altitude scale. To this end, we begin with the ideal gas law:

(3) $p(z) = \rho(z) \cdot R_d \cdot T_v(z)$,

where p(z) is the pressure, ρ (z) is the density, R_d is the gas constant for dry air (287.05307 J /kg K), and $T_v(z)$ is the virtual temperature. T_v is derived by re-writing the ideal gas law for dry air in a manner which accounts for water vapor, thus allowing the gas constant for dry air to be used. The resulting expression is:

(4) $T_v(p, T, RH) = \frac{T}{1 - \frac{RH \cdot e_s(T)}{100 \cdot p} \cdot (1 - \varepsilon_a)}$.

T_v is a function of pressure (p), temperature (T), and relative humidity (RH), and in the form given here also involves the saturation vapor pressure for water vapor (e_s), which a function of only T, and the ratio of the molecular weights of wet and dry air (ε_a = 0.622). T_v accounts for the fact that air containing water vapor is less dense than dry air, and it is always greater than the actual physical temperature. If the virtual temperature were not used, the gas constant would be a function of the vapor content (that is, it wouldn't be constant).

Now the differential form of the hydrostatic equation is:

(5) $\frac{d}{dz} p(z) = -\rho(z) \cdot \gamma(z,\phi)$,

where ρ (z) is the density and $\gamma(z,\phi)$ is gravity. If we use the ideal gas law (equation (3)) to eliminate ρ (z), we obtain the differential form:

(6) $\gamma(z,\phi)\, dz = -R_d\, T_v(z)\, d\,[\ln(p(z))]$

Now referring to equation (1) for the gravitational potential energy, it is immediately recognized that the integral of the left hand side of equation (6) is the gravitational potential energy per unit mass of air. On dividing equation (6) by γ_{45} and integrating from $z = Z_1$ to $z = Z_2$, we arrive at an expression for calculating the geopotential height between any two radiosonde levels without knowing anything about the local gravity.

(7) $H_{12}(Z,\phi) = \frac{\Phi_{12}(Z,\phi)}{\gamma_{45}} = \frac{1}{\gamma_{45}} \cdot \int_{Z_1}^{Z_2} \gamma(z,\phi)\, dz = \frac{-R_d}{\gamma_{45}} \cdot \int_{Z_1}^{Z_2} T_v(z)\, d(\ln(p(z))) = \frac{R_d}{\gamma_{45}} \cdot Tv_{avg} \cdot \ln\left(\frac{p_1}{p_2}\right)$,

The right-most expression is known as the *hypsometric equation* ; it is important since it relates the thickness, $H_{12}(Z,\phi)$, between two pressure surfaces (p_1 and p_2) directly to the average virtual temperature (Tv_{avg}) (and two constants: R_d and γ_{45}). The *hypsometric equation* can be summed between radiosonde pressure levels from the surface to the highest measurement to obtain the geopotential height at any radiosonde level. This of course can only be done from the surface, which generally will not be at sea level. However, the WMO provides the geopotential height for every radiosonde launch site, so this offset can be added to the sum to get the geopotential height relative to the geoid, or mean sea level.

Finally, we address the issue of converting between geopotential height and geometric height. This can be done by writing the expression for the geopotential height in differential form:

(8) $\gamma_{45}\, dh = \gamma(z,\phi)\, dz$.

In fact, this expression is often used as the definition of the geopotential height. Denoting for simplicity R as the radius of the earth at latitude ϕ, and using the inverse square law for gravity, we can write for any latitude (ϕ) and geometric altitude (z):

(9) $\gamma(R + z, \phi) = \gamma(R, \phi)\, (R / (R + z))^2$.

Substituting equation (9) into equation (8) and integrating from z = 0 to z = Z -- assuming $\gamma(z,\phi)$ does not vary with z -- gives:

(10) $H(Z,\phi) = \frac{\gamma(Z,\phi)}{\gamma_{45}} \cdot \frac{R \cdot Z}{R + Z}$

Now of course $\gamma(z,\phi)$ does vary with geometric altitude (z) so this must be dealt with. A trick suggested by W. D. Lambert in 1949 in the **Smithsonian Meteorological Tables (SMT),** which is a frequently cited source for converting between geometric altitude and geopotential height, was to compensate for the variation of $\gamma(z,\phi)$ by letting R assume an appropriate value that satisfied certain boundary conditions. This would account for the fact that equation (9) only applied to a non-rotating sphere composed of spherical shells of equal density, which is not true for earth, and for the fact that the centrifugal component of gravity increases linearly with radius, rather than inversely with the square of the radius as shown in this equation. Taking the derivative of equation (9) with respect to z, evaluating the resulting expression at z = 0, and then solving for R, we obtain:

(11) $R_{SMT}(\phi) = \frac{2\gamma_{SMT}(\phi)}{-\frac{d}{dz}(\gamma_{SMT}(\phi))}$

Note the minus sign in the denominator of equation (11), and remember that the derivative is evaluated at z=0 (which is why the z dependence is dropped on γ_{SMT}). The notation R_{SMT} is used because this technique was originally described in the **Smithsonian Meteorological Tables** (R. J. List, Editor, 1968). Note that R_{SMT} is not the radius of the Earth at any particular latitude; rather it is a value that is needed to account for the combined effect of gravitational and centrifugal forces with altitude z. Using this notation, equation (10) becomes:

(12) $H_{SMT}(Z,\phi) = \frac{\gamma(Z,\phi)}{\gamma_{45}} \cdot \frac{R_{SMT}(\phi) \cdot Z}{R_{SMT}(\phi) + Z}$,

or on inverting to solve for Z

(13) $Z_{SMT}(H,\phi) = \frac{R_{SMT}(\phi) \cdot H}{\frac{\gamma(Z,\phi)}{\gamma_{45}} \cdot R_{SMT}(\phi) - H}$.

Since the later equation is often used in calculations to convert radiosonde geopotential height to geometric altitude, a Taylor series expansion can be used over radiosonde altitudes to obtain:

(14) $Z_{SMT}(H,\phi) = \left(1 + 2.373 \cdot 10^{-3} \cdot \cos(2\phi)\right) \cdot H + \left(1 + 8.6476 \cdot 10^{-3} \cdot \cos(2\phi)\right) \cdot \frac{H^2}{6356.6818 \cdot km}$

where H is in km and we used $\gamma_{45} = 9.80665\ m/s^2$. This equation would agreed exactly with the **Smithsonian Meteorological Tables** if we had used $\gamma_{45} = 9.80\ m/s^2$. There are some issues however with using the **Smithsonian Meteorological Tables** that go beyond the fact that we have computers now that can easily calculate the conversion between geometric and geopotential height. The tables use the **International Ellipsoid 1935** (which is different from **WGS-84**) and they assume γ_{45} = 9.8 m s^{-2} exactly. But there are other issues. The value of normal gravity is stated to be:

(15) $\gamma_{SMT}(\phi) := 9.806160 \cdot \left(1 - 0.0026373 \cdot \cos(2\cdot\phi) + 0.0000059 \cdot \cos(2\cdot\phi)^2\right) \cdot m \cdot s^{-2}$

and is based on a 1949 report of the International Association of Geodesy titled: "*Gravity Formulas for Meteorological Purposes* " by W. D. Lambert. The tables also use the following expression for the negated derivative of normal gravity:

(16) $-\frac{d}{dz}(\gamma_{SMT}(\phi)) = 3.085462 \cdot 10^{-6} + 2.27 \cdot 10^{-9} \cdot \cos(2\cdot\phi) - 2 \cdot 10^{-12} \cdot \cos(4\cdot\phi)$

Unfortunately, the origin of this expression has not been published (W. D. Lambert, *Some notes on the calculation of the geopotential, unpublished manuscript* , 1949), so the assumptions going into it are not clear. In any event, if the value of **R** found in equation (11) is used in equation (10), it satisfies two boundary conditions: it has the correct surface gravity, and it has the correct vertical gravity gradient based on the **International Ellipsoid of 1935**. Lest there be any doubt that this is a fictitious radius, it is easily shown that the radius $\mathbf{R_{SMT}}$ is greater at the pole than at the equator, and very different from any real earth radius (see **Figure 3**).

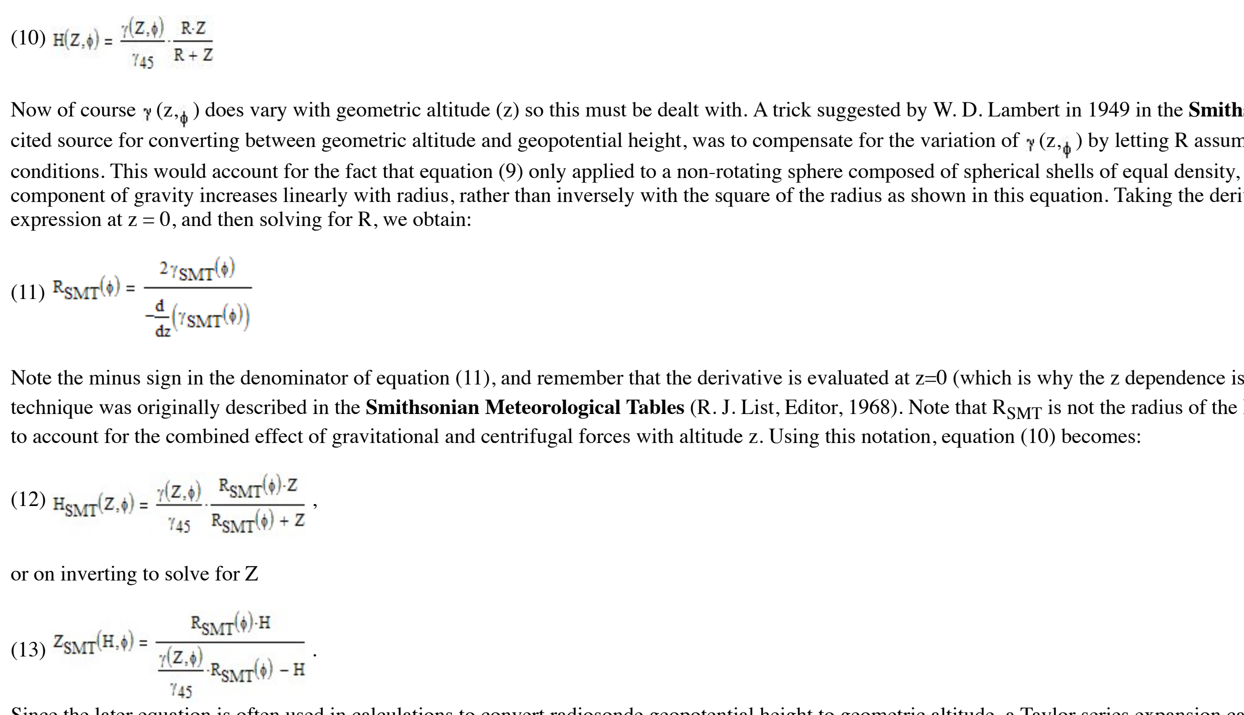


Figure 3. The ficticious "radius" ginned up by W. D. Lambert (**Rsmt**) behaves in a completely different manner from the ellipsoidal radius of the earth (**Rellipsoid**) and can they can differ by >40 km, yet many papers and online discussions mistakenly refer to it as the "radius of the earth at a particular latitude." **Rwgs** is a closed-form version of the ficticious radius, which is derived below. **Russa** is the ficticious radius (based on the **Smithsonian Meteorological Tables** with g = 9.80665 m/s2) at a latitude of 45.542 degrees assumed for the **US Standard Atmosphere (1976)** (see discussion below). **Rsmt** and **Rwgs** differ by almost 350 m.

Rather than work with this undocumented result of W. D. Lambert, I have followed the same approach using the currently recognized ellipsoid of revolution, namely **WGS-84**. Before proceeding, we need to specify a number of additional derived parameters for this earth reference system. They include:

Semi-minor axis (b)	6356.7523142 km
Flattening (f = (a - b) / a)	0.003352811
Linear eccentricity ($E := \sqrt{a^2 - b^2}$)	521.854008974 km
Eccentricity (e= E / a)	0.081819
Polar gravity (γ_p)	9.8321849378 m s^{-2}
Equatorial gravity (γ_e)	9.7803253359 m s^{-2}
Somigliana's Constant ($k_s := \frac{b}{a} \cdot \frac{\gamma_p}{\gamma_e} - 1$)	1.931853×10^{-3}
Gravity ratio ($m_r := \frac{\omega^2 \cdot a^2 \cdot b}{GM}$)	0.003449787

Armed with these values, we begin by writing Somigliana's Equation for normal gravity on the surface of an ellipsoid of revolution (Heiskanen and Moritz, Physical Geodesy, 1969):

(17) $\gamma_s(\phi) := \gamma_e \cdot \left(\frac{1 + k_s \cdot \sin(\phi)^2}{\sqrt{1 - e^2 \cdot \sin(\phi)^2}}\right)$

On performing a Taylor series expansion in the vertical, we obtain the following expression for the normal gravity:

(18) $\gamma(z,\phi) := \gamma_s(\phi) \cdot \left[1 - 2 \cdot \left(1 + f + m_r - 2 \cdot f \cdot \sin(\phi)^2\right) \cdot \frac{z}{a} + 3 \cdot \left(\frac{z}{a}\right)^2\right]$,

or numerically:

(19) $\gamma(z,\phi) = \gamma_s(\phi) \cdot \left[1 - 2 \cdot \left(1.006803 - 0.006706 \cdot \sin(\phi)^2\right) \cdot \frac{z}{a} + 3 \cdot \left(\frac{z}{a}\right)^2\right]$.

On taking the derivative of this expression with respect to z and evaluating it at the surface, we obtain the following version of equation (11):

(20) $R(\phi) := \frac{a}{1 + f + m_r - 2 \cdot f \cdot \sin(\phi)^2}$,

or numerically:

(21) $R(\phi) = \frac{6378.137}{1.006803 - 0.006706 \cdot \sin(\phi)^2} \cdot km$

This expression can then be used in equation (10) to convert from geometric altitude to geopotential height; that is:

(22) $H(Z,\phi) = \frac{\gamma_s(\phi)}{\gamma_{45}} \cdot \frac{R(\phi) \cdot Z}{R(\phi) + Z}$

$H(Z,\phi)$ calculated using this expression agrees with direct integration of the value of gravity given by equation (18) to within 2 mm at the equator at 20 km! Furthermore, if a more accurate expression is used for gravity , instead of the Taylor series expansion given in equation (18), the agreement is better than 1.5 mm at all latitudes at 20 km! As noted above, it is often necessary to invert equation (22) to solve for Z; that is:

(23) $Z(H,\phi) = \frac{R(\phi) \cdot H}{\frac{\gamma_s(\phi)}{\gamma_{45}} \cdot R(\phi) - H}$.

If equation (23) is approximated by a Taylor series expansion, and optimized for altitudes from 0 to 25 km, we have:

(24) $Z(H,\phi) := \left(1 + 0.002644 \cdot \cos(2\cdot\phi)\right) \cdot H + \left(1 + 0.0089 \cdot \cos(2\cdot\phi)\right) \cdot \frac{H^2}{6245 \cdot km}$

to an accuracy of 20 cm, the best accuracy being near the ground and at ~20 km.

As an aside, it is worth noting that the **US Standard Atmosphere 1976** provides the following expressions for the geometric/geopotential height conversion:

(25) $H(Z) = R_o\, Z / (R_o + Z)$,

or, on solving for Z,

(26) $Z(H) = R_o\, H / (R_o - H)$.

where R_o = 6356.766 km. Since the **US Standard Atmosphere** is nominally defined for a latitude of 45 degrees, and no other, these equations are trivially derived from equations (22) and (23) by assuming $\gamma_s(\phi) = \gamma_{45}$ = 9.80665 m/s^2. As mentioned earlier, gravity is 9.80665 m/s^2 at a latitude of 45.542 degrees, and the value of R_o used in the US Standard Atmosphere is the ficticious radius at this latitude (not 45 degrees).

Formulae are also provided in the **Federal Meteorological Handbook No. 3**, which discusses requirements for rawindsonde and pibal observations (Office of the Federal Coordinator of Meteorology, 1997). Regrettably, this primary reference uses equation (15) for $\gamma(\phi)$, which is based on a 1935 reference ellipsoid, and it states that the relevant radius in equation (22) is "*Re, the radius of the earth at latitude* ϕ ." As explained above, this is incorrect. I have attempted several times to contact the OFCM and my correspondence has not been acknowledged. Not surprisingly I have found many web sites with conversion calculators that blindly use this incorrect information. Caveat Emptor!

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